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Time-Optimal Spin-Up Maneuvers for Flexible Spacecraft

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I. Introduction

In recent years, time-optimal rest-to-rest rotational maneuvers for flexible spacecraft have been thoroughly investigated.[1] In particular, a system consisting of a rigid hub controlled by a single actuator, with one or more elastic appendages attached to the hub, was studied by several researchers,[2,3] who investigated the properties of the resulting bang-bang solution. It has been shown that, for undamped systems, the optimal control function possesses the skew symmetric property

$$u(t) = -u(\tilde{t} - t) \tag{1}$$

where $\tilde{t}$ is the optimal time. Only recently was this property exploited to demonstrate that for the one-bending mode case the maximal number of switches is three.[4]

The purpose of this Note is to study time-optimal spin-up maneuvers for flexible spacecraft. The motivation for this research is twofold: First, spacecraft often carry out this type of maneuver to become spin stabilized in space, and second, the analysis manifests the fact that properties of optimal control, such as the maximal number of switches, can vary considerably by making a simple change in the boundary conditions.

II. Problem Formulation

We consider the time-optimal single-axis spin-up maneuvering problem for a system consisting of a rigid hub with one or more undamped elastic appendages attached to it. The system is controlled by a single actuator that exerts an external torque on the rigid hub. The discretized dynamic equations can be shown to have the following form[2,3]:

$$\dot{x}(t) = \begin{bmatrix} 0 & & & & & \\ & 0 & 1 & & & \\ & -\omega_1^2 & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ & & & & -\omega_k^2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} b_0 \\ 0 \\ b_1 \\ \vdots \\ 0 \\ b_k \end{bmatrix} u(t) \tag{2}$$

where the state vector is denoted by $x \equiv [x_0, x_1, x_2, \ldots, x_{2k-1}, x_{2k}]^T$.

Notice that the first equation corresponds to the rigid mode (x_0 being the angular velocity), whereas the rest are equations for the first k flexible modes. We pose the following optimal control problem: find the minimum time $\tilde{t}$ and the corresponding time-optimal control $\tilde{u}(\cdot) : [0, \tilde{t}] \to [-1, 1]$ that drives the system from initial conditions at the origin to the following final target set:

$$x(\tilde{t}) = [p \quad 0 \quad 0 \quad \cdots \quad 0]^T \tag{3}$$

where p is given and for simplicity we will assume that $p \geq 0$. The extension to negative p is straightforward. The elastic energy, therefore, is zero at both boundary points, and the work done by the control is completely transformed to rigid-mode kinetic energy.

III. Problem Analysis

Because the system (2) is controllable by a single control, there exists a unique bang-bang solution to the problem.

Recall that for rest-to-rest problems the control is skew symmetric with respect to $\tilde{t}/2$ [Eq. (1)]. The following theorem asserts that the spin-up optimal control is symmetric with respect to $\tilde{t}/2$.

Theorem 1: Let $\tilde{u}(\cdot) : [0, \tilde{t}] \to [-1, 1]$ be the time-optimal control steering the state vector x from the origin to $x(\tilde{t})$, then

$$\tilde{u}(t) = \tilde{u}(\tilde{t} - t) \tag{4}$$

Proof: We shall use the following lemma.

Lemma: Consider an auxiliary problem of finding the time-optimal control for the system

$$\dot{y}_0(t) = -b_0 u(t), \qquad \dot{y}_i(t) = -y_{i+1}(t)$$
$$y_{i+1}(t) = \omega^2_{(i+1)/2} y_i(t) - b_{(i+1)/2} u(t)$$
$$i = 1, 3, 5, \ldots, 2k - 1 \tag{5}$$

which steers $\mathbf{y}$ from the origin to a target set $y_f = [-p, 0, 0, \ldots, 0, 0]^T$.

Then, the optimal solution to our original problem $\tilde{u}(\cdot) : [0, \tilde{t}] \to [-1, 1]$ is also the optimal solution to the auxiliary problem.

Proof of the Lemma: Assume an optimal control function $\tilde{u}(\cdot)$ driving the system (2) while satisfying Eq. (3), and apply it to Eq. (5) with zero initial conditions. Clearly the terminal condition for y_0 [namely, $y_0(\tilde{t}) = -p$] is met. Notice also that any ith pair of equations (for $i = 1, 3, 5, \ldots, 2k - 1$) can be written as a second-order equation

$$\ddot{y}_i(t) + \omega^2_{(i+1)/2} y_i(t) = b_{(i+1)/2} \tilde{u}(t) \tag{6}$$

Identically, in the original problem we have

$$\ddot{x}_i(t) + \omega^2_{(i+1)/2} x_i(t) = b_{(i+1)/2} \tilde{u}(t) \tag{7}$$

Thus, the zero terminal conditions for y_i and y_{i+1} at $t = \tilde{t}$ are met if, and only if, the terminal values of x_i and x_{i+1} are zeros. Therefore any feasible control that satisfies the terminal conditions for the main problem also satisfies the required conditions of the auxiliary problem and vice versa.

Assume now that the time-optimal control for the auxiliary problem is $\tilde{\tilde{u}}(\cdot) : [0, \tilde{\tilde{t}}] \to [-1, 1]$; hence, $\tilde{\tilde{t}} \leq \tilde{t}$ {because $\tilde{u}(\cdot) : [0, \tilde{t}] \to [-1, 1]$ is a feasible control to this problem}. On the other hand, because $\tilde{\tilde{u}}(\cdot)$ drives Eq. (2) to the required target set in time $\tilde{\tilde{t}}$, we conclude from the optimality of $\tilde{u}(\cdot)$ that $\tilde{t} \leq \tilde{\tilde{t}}$, and thus $\tilde{t} = \tilde{\tilde{t}}$. Because the optimal control is unique for both problems, $\tilde{\tilde{u}}(\cdot) = \tilde{u}(\cdot)$, and the Lemma is proved.

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We now return to the original problem, and we let $\tau = \tilde{t} - t$ to obtain

$$\frac{\mathrm{d}x_0(\tau)}{\mathrm{d}\tau} = -b_0 u(\tau), \qquad \frac{\mathrm{d}x_i(\tau)}{\mathrm{d}\tau} = -x_{i+1}(\tau)$$
$$\frac{\mathrm{d}x_{i+1}(\tau)}{\mathrm{d}\tau} = \omega^2_{(i+1)/2} x_i(\tau) - b_{(i+1)/2} u(\tau) \tag{8}$$

where $i = 1, 3, 5, \ldots, 2k-1$. Our spin-up problem can be reformulated using Eq. (8) as follows. Find $\hat{\tau}$ and $\hat{u}(\cdot) : [0, \hat{\tau}] \to [-1, 1]$ that steers system (8) from

$$x(\tau)|_{\tau=0} = [p \quad 0 \quad 0 \quad \cdots \quad 0]^T$$

to

$$x(\tau)|_{\tau=\hat{\tau}} = [0 \quad 0 \quad 0 \quad \cdots \quad 0]^T$$

while minimizing $\hat{\tau}$.

Clearly, due to the identity of Eqs. (2) and (8) (up to a change of variables), we find that

$$\hat{u}[\tau(t)] = \tilde{u}(t) \tag{9}$$

We now make the following change of variables:

$$\hat{x}_0 \equiv x_0 - p, \qquad \hat{x}_i \equiv x_i \qquad i = 1, \ldots, 2k$$

Equation (8) remains the governing equation for $\hat{x}$, and the optimal control $\hat{u}(\cdot) : [0, \hat{\tau}] \to [-1, 1]$ drives the newly defined states $[\hat{x}_0, \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{2k-1}, \hat{x}_{2k}]^T$ from the origin to $\hat{x}(\tau)|_{\tau=\hat{\tau}} = [-p \ 0 \ 0 \ \cdots \ 0]^T$.

We arrive at a problem formulation identical to the auxiliary problem (with τ rather than t as the independent variable). From the Lemma we have $\hat{\tau} = \tilde{t}$ and

$$\hat{u}(\cdot) = \tilde{u}(\cdot)$$

or, in particular,

$$\hat{u}[\tau(t)] = \tilde{u}[\tau(t)]$$

From the preceding and Eq. (9), we conclude that

$$\tilde{u}[\tau(t)] = \tilde{u}(t)$$

or

$$\tilde{u}(\tilde{t} - t) = \tilde{u}(t) \qquad \square$$

We note that the property of symmetry is a direct result of the time-reversal system dynamics. Adding friction will make the system nonreversal in time and will break the symmetry of the solution.

To further characterize the optimal control, we shall employ the minimum principle, which, for this problem, is a necessary and sufficient condition for optimality.[5]

We define the Hamiltonian function

$$H(x, \lambda, u) = 1 + \lambda_0 b_0 u + \sum \lambda_i x_{i+1} + \sum \lambda_{i+1}\left(-\omega^2_{(i+1)/2} x_i + b_{(i+1)/2} u\right)$$

[The summation is over all values of i, namely, $(1, 3, 5, \ldots, 2k-1)$.]

The corresponding adjoint system is as follows:

$$\dot{\lambda}_0(t) = 0, \qquad \dot{\lambda}_i(t) = \omega^2_{(i+1)/2}\lambda_{i+1}(t), \qquad \dot{\lambda}_{i+1}(t) = -\lambda_i(t) \tag{10}$$

where $i = 1, 3, 5, \ldots, 2k-1$. The minimum principle requires that

$$\tilde{u}(t) = \arg\min_{|u| \le 1} H[x(t), \lambda(t), u] \tag{11}$$

Hence

$$\tilde{u}(t) = -\mathrm{sgn}\left[b_0\lambda_0(t) + \sum b_{(i+1)/2}\lambda_{i+1}(t)\right] \equiv -\mathrm{sgn}[\sigma(t)] \tag{12}$$

Singular arcs are excluded due to the normality of the system.

The solution to Eq. (10) can be easily obtained as

$$\lambda_0(t) = \lambda_0(0)$$
$$\lambda_i(t) = \lambda_{i+1}(0)\omega_{(i+1)/2}\sin\left(\omega_{(i+1)/2}t\right) + \lambda_i(0)\cos\left(\omega_{(i+1)/2}t\right) \tag{13}$$
$$\lambda_{i+1}(t) = \lambda_{i+1}(0)\cos\left(\omega_{(i+1)/2}t\right) - \frac{\lambda_i(0)}{\omega_{(i+1)/2}}\sin\left(\omega_{(i+1)/2}t\right)$$

where $i = 1, 3, 5, \ldots, 2k-1$. The solution of the two-point boundary-value problem requires us to find the initial value for the adjoint vector $\lambda(0)$ such that the resulting control [Eq. (12)] will steer the system (2) to the terminal condition (3). Integrating (2) is also straightforward because the control is a superposition of step functions. Thus, assuming that the control switches at $\{t_1, t_2, \ldots, t_m\}$ (not counting $t = 0$ as a switching point), we obtain

$$x_0(\tilde{t}) = b_0\left[\tilde{t} + 2\sum_{j=1}^{m}(-1)^j(\tilde{t} - t_j)\right]$$
$$x_i(\tilde{t}) = \omega^{-2}_{(i+1)/2} b_{(i+1)/2}\left(\left[1 - \cos\left(\omega_{(i+1)/2}\tilde{t}\right)\right] + 2\sum_{j=1}^{m}(-1)^j\left\{1 - \cos\left[\omega_{(i+1)/2}(\tilde{t} - t_j)\right]\right\}\right) \tag{14}$$
$$x_{i+1}(\tilde{t}) = \omega^{-1}_{(i+1)/2} b_{(i+1)/2}\left\{\sin\left(\omega_{(i+1)/2}\tilde{t}\right) + 2\sum_{j=1}^{m}(-1)^j\sin\left[\omega_{(i+1)/2}(\tilde{t} - t_j)\right]\right\}$$

where $i = 1, 3, \ldots, 2k-1$. In Ref. 3 it was proposed to use solutions of Eq. (14) that minimize $\tilde{t}$ and to evaluate $\lambda(0)$ from the switching condition to verify the global optimality.

In the sequel we will restrict the discussion to one flexible mode. We have the following result.

Theorem 2: The minimal number of switches for one flexible mode system is

$$\bar{m} = 2\,\mathrm{Int}[\Omega p / 2\pi b_0] + 2 \tag{15}$$

where Ω is the single flexible mode frequency, and $\mathrm{Int}[z]$ denotes the largest integer that is less than z.

Remarks: 1) In the parallel situation for the rest-to-rest maneuver, the maximal number of switches is three[4]! 2) Note that $\mathrm{Int}[\Omega p/2\pi b_0]$ is the number of complete cycles of oscillation of the appendages in the time $\tilde{t}_{\min} = p/b_0$ (the rigid mode requirement). 3) Notice that, by the symmetry of Theorem 1, the number of switches is always even (not only for the single case mode).

Proof: The switching function takes the form

$$\sigma(t) = b_0\lambda_0(0) + b_1\left[\lambda_2(0)\cos(\Omega t) - \frac{\lambda_1(0)}{\Omega}\sin(\Omega t)\right]$$
$$= c_1 + c_2\cos(\Omega t + \delta) \tag{16}$$

Notice that a zero-switching control cannot satisfy the boundary condition (3). Hence, by symmetry (Theorem 1), we require the existence of at least two switching points. Moreover, if t_s is a switching point, then so is $t_s + (2\pi/\Omega)$, and an additional switching point exists between the two, as illustrated in Fig. 1 (except under some

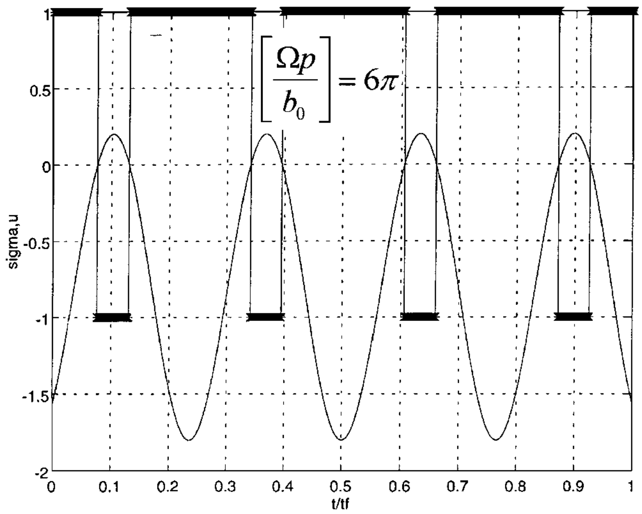

Fig. 1 Optimal control and the switching function.

isolated conditions). As already stated, the rigid mode by itself requires $\tilde{t}_{\min} = p/b_0$. Because $\tilde{t} \geq \tilde{t}_{\min}$, the minimum number is as stated by Eq. (15). □

Denote by α the phase of the first switching point (and, hence, the phase-to-go of the last) and by φ the phase difference between the first two switching points (and, hence, the last two), or equivalently $t_1 = \alpha/\Omega$, and $t_2 = (\alpha + \varphi)/\Omega$. Also denote p' as the quantity $[\Omega p/b_0]$.

We have shown that all the switching points can be traced back from the last two by multiple shifts of complete periods. Thus from Eq. (14) we obtain (using $t_1 = \alpha/\Omega$, $t_2 = (\alpha + \varphi)/\Omega$, and the periodic properties of the trigonometric functions)

$$x_0(\tilde{t}) = (b_0/\Omega)\{2\alpha - \varphi + [(m/2) - 1](2\pi - 2\varphi)\} = p$$

$$x_1(\tilde{t}) = \Omega^{-2} b_1\{[1 - \cos(2\alpha + \varphi)] + m[\cos(\alpha + \varphi) - \cos\alpha]\} = 0$$

$$x_2(\tilde{t}) = \Omega^{-1} b_1\{\sin(2\alpha + \varphi) + m[-\sin(\alpha + \varphi) + \sin\alpha]\} = 0 \tag{17}$$

Although we have three equations and three unknowns α, φ, and m, one may wonder how can we expect to find a solution with the restriction that m is an integer! The answer to this puzzle is that the last two expressions in Eq. (17) are equivalent to each other as asserted by the next theorem.

Theorem 3: The conditions

$$\{[1 - \cos(2\alpha + \varphi)] + m[\cos(\alpha + \varphi) - \cos\alpha]\} = 0$$
$$\{\sin(2\alpha + \varphi) + m[-\sin(\alpha + \varphi) + \sin\alpha]\} = 0 \tag{18}$$

are equivalent to each other.

Proof: Notice that Eq. (18) can be written down as

$$\mathrm{Re}(A) + m\,\mathrm{Re}(B) = 0, \qquad \mathrm{Im}(A) + m\,\mathrm{Im}(B) = 0 \tag{19}$$

where $A = 1 - e^{-j(2\alpha+\varphi)}$ and $B = e^{-j(\alpha+\varphi)} - e^{-j\alpha}$. From Fig. 2 it is clear that A and B are collinear, and hence $\mathrm{Im}(A) + m\,\mathrm{Im}(B) = 0$ if, and only if, $\mathrm{Re}(A) + m\,\mathrm{Re}(B) = 0$. □

Remarks: 1) From the proof of the theorem and Fig. 2, it is clear that requiring

$$|A| - m|B| = 0 \quad \text{or} \quad m\sin(\varphi/2) - \sin[\alpha + (\varphi/2)] = 0 \tag{20}$$

is also equivalent to Eq. (18). Notice that because $m \geq 2$, we require that $\sin(\varphi/2) \leq \frac{1}{2}$; thus, $\varphi \leq (\pi/3)$ or $\varphi \geq 2\pi - (\pi/3)$.

2) Each admissible φ results in a finite set of solutions whose number may vary from a single trajectory [when the only solution to Eq. (20) is with $m = 2$] to as many solutions as Eq. (20) can provide.

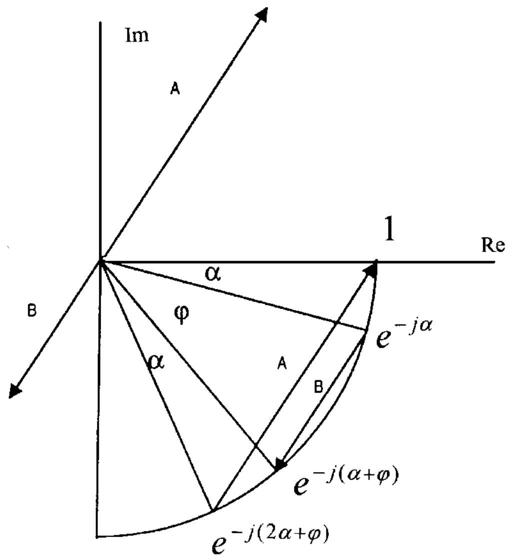

Fig. 2 Geometric picture of Eq. (19).

Each of these trajectories will terminate with zero elastic energy and at a final p' according to

$$p' = \{2\alpha - \varphi + [(m/2) - 1](2\pi - 2\varphi)\} \tag{21}$$

It remains to prove that every solution to Eqs. (20) and (21) is an optimal solution. This will be done in following theorem.

Theorem 4: For all triples $\{\varphi, \alpha, m\}$ satisfying Eqs. (20) and (21), the control constructed by the preceding given switching rules is optimal.

Proof: By construction we switch at $t_1 = \alpha/\Omega$ and $t_2 = (\alpha + \varphi)/\Omega$, and then at every complete period following t_1 and t_2; thus, if

$$\sigma(t_1) = b_0\lambda_0(0) + b_1[\lambda_2(0)\cos(\alpha) - (\lambda_1(0)/\Omega)\sin(\alpha)] = 0$$

$$\sigma(t_2) = b_0\lambda_0(0) + b_1[\lambda_2(0)\cos(\alpha + \varphi) - (\lambda_1(0)/\Omega)\sin(\alpha + \varphi)] = 0 \tag{22}$$

then the rest of the switching points are automatically satisfied. From Eq. (22) we find that $\lambda(0)$ is as follows:

$$\lambda(0) = \lambda_0(0)\begin{bmatrix} 1 \\ \dfrac{\Omega b_0}{b_1}\dfrac{\cos\alpha - \cos(\alpha + \varphi)}{\sin\varphi} \\ \dfrac{b_0}{b_1}\dfrac{\sin\alpha - \sin(\alpha + \varphi)}{\sin\varphi} \end{bmatrix} \tag{23}$$

Requiring $H[x(0), \lambda(0), u(0)] = 0$ will determine uniquely the initial costate vector. Thus there is a unique solution satisfying the minimum principle. Because for these types of problems the minimum principle is necessary and sufficient for optimality,[5] the proof is complete. □

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